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Proportional Reinsurance is based on Maximizing the Lundberg Exponent

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ABSTRACT: Optimizing risk in the insurance market, particularly in reinsurance, is a major concern due to potential risks in a rapidly developing industry. Providing evaluation criteria for reinsurance models is crucial for insurance companies to enhance business performance. Risk management in reinsurance is of great significance in the insurance business and has been extensively researched. Many different reinsurance models can be used such as the mean criterion, the expectation-variance criterion, the bankruptcy probability function criterion, and the Lundberg exponent optimization criterion. This paper focuses on studying risk optimization in proportional reinsurance by maximizing the Lundberg exponent of the Cramer-Lundberg risk model to minimize the probability of bankruptcy $\psi(x)$. At the same time, we provide the algorithm to determine the Lundberg exponent based on the scaling factor, and risk optimization is evaluated accordingly.

KEYWORDS: Optimizing risk, probability of bankruptcy, proportional reinsurance, reinsurance, Lundberg exponent.

I. INTRODUCTION

Reinsurance is a crucial step in the business cycle that aims to distribute risks and ensure the survival of insurance companies and the insurance market. To assess the risk involved in the reinsurance model, we need to gather statistics on the company's insurance data, construct a distribution function for insurance payments, and establish evaluation criteria. The mathematical model of the reinsurance method then evaluates these criteria based on the payment distribution function, providing optimal parameter values. Depending on their investment portfolio, insurance companies select evaluation criteria and parameter values for the model to effectively manage risk.

It is vital for insurance and reinsurance companies to maintain capital levels above a certain threshold at all times. The concept of bankruptcy time refers to the earliest point at which the capital falls below this threshold. The probability of this occurrence (going bankrupt), known as the bankruptcy probability, is a critical consideration in risk management and optimization in reinsurance. The goal is to find a reinsurance plan that minimizes the probability of bankruptcy. Reinsurance, therefore, serves as insurance for the risks that insurers must bear. It involves transferring a portion of the responsibility, along with a portion of the insurance cost, to another insurer through a reinsurance contract. Reinsurance is closely tied to original insurance operations, as it is built upon them. It provides psychological security for insurers, balances insurance services, safeguards them against major catastrophic incidents, and ensures financial stability. However, reinsurance also entails shifting some, if not most, of the insurance costs to the reinsurer. Consequently, reinsurance can significantly impact the financial indicators of insurance companies, either positively or negatively.

The optimal reinsurance problem is typically not approached by directly minimizing the ruin probability itself. This is because the ruin probability does not have an explicit expression in most situations, even in the classical risk model. Therefore, some scholars choose to minimize the Lundberg upper bound of the ruin probability as an alternative value function. In such cases, a possible reinsurance arrangement maximizes the Lundberg exponent. In the Cramér-Lundberg risk model, the ruin probability has a simple exponential upper bound known as the Lundberg upper bound. Consequently, the Lundberg exponent serves as an alternative risk measure for determining a company's solvency capability over a long-term horizon. The optimal reinsurance problem has been extensively studied in terms of maximizing the Lundberg exponent over the past few decades. For instance, In the paper *"On the maximisation of the adjustment coefficient under proportional reinsurance"* Hald & Schmidli (2004) examined the optimal proportional reinsurance problem for an insurer using the Cramér-Lundberg model and a general renewal risk model, respectively.

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Research has shown how to maximize the adjustment coefficient in the case of proportional reinsurance. Schmidli (2002a) investigated optimal excess-of-loss reinsurance under the Cramér-Lundberg model. Meng et al. (2023) investigated the multiple per-claim reinsurance based on maximizing the Lundberg exponent, research has shown that, based on maximization of the insurer's Lundberg exponent, the optimal reinsurance is formulated within a static setting, In general, these optimal strategies are shown to have non-piecewise linear structures, differing from conventional reinsurance strategies such as quota-share, excess-of-loss, or linear layer reinsurance arrangements. Until now, the majority of authors have concentrated on specific forms of reinsurance when developing risk measurement and assessment techniques. These methods include assessment based on average value criteria, assessment based on variance criteria, and evaluation based on the non-bankruptcy probability function criterion. In this paper, we utilize an evaluation approach that focuses on maximizing the Lundberg exponent is based on a one-parameter proportional reinsurance model

II. THEORETICAL FRAMEWORK

A. Risk measurement and assessment methods.

In this section, we will provide a brief overview of risk assessment methods in the reinsurance model. These methods include the mean value criterion, the variance-expectation criterion, and the probability of non-bankruptcy function. Special was the maximizing of the Lundberg exponent, detailed explanations, and algorithms will be presented in the results section (Lucas et al., 2018).

1) Evaluation is based on average value criteria:

The average value of the random variable X is a real number, denoted as $E(X)$, and is determined as follows (Шон, 2005):

If X is a discrete random variable with probability distribution $P(X = x_k) = p_k$ then:

$$E(X) = \sum_{i=1}^{\infty} x_i P(X = x_i) = \sum_{i=1}^{\infty} x_i p_i$$

If X is a continuous random variable with a density function $f_X(x)$, then:

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x) dx$$

If X is a continuous random variable with a cumulative distribution function $F(x)$, then:

$$E(X) = \int_{-\infty}^{+\infty} (1 - F(x)) dx$$

2) Evaluation is based on variance criteria:

The variance of the random variable X is a real number, denoted $D(X)$ determined:

$$D(X) = E(X - E(X))^2$$

Implementing the right side we have:

$$D(X) = E[X^2 - 2.X.E(X) + (E(X))^2] = E(X^2) - (E(X))^2$$

The variance of a random variable is used to characterize the degree of dispersion of the values of that random variable around its mean value.

If X is a non-negative real-valued random variable with $E(X) < \infty$ we have:

$$E(X^2) = 2\int_{0}^{+\infty} x.P(X > x) dx$$

In case the random variable X has a cumulative distribution function $F(x)$ then we have:

$$E(X^2) = 2\int_{0}^{+\infty} x(1 - F(x)) dx$$

3) Evaluation is based on the criteria of probability function of not going bankrupt:

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Symbol:

$F_{X_i}(y)$, $0 \le y < \infty$ - Distribution function of the size of payments;

X_i; $m_k = \mathbf{M}^{(k)} X_i < \infty$, $k \ge 1$ - The Kth mathematical expectation of the distribution function; $C(t)$ - Value of the amount of money collected from the policyholder during a period of time $[0,t)$;

$x \ge 0$ - Initial investment capital of the insurance company.

Without loss of generality, assume $F_{X_i}(y)$ is a continuous function and $F_{X_i}(0) = 0$. Thus, during its operation, the insurance company's capital at time t is determined as follows:

$$Z(t) = x + C(t) - \sum_{i=1}^{N(t)} X_i. \qquad (1)$$

In which, $N(t)$ - number of times to pay in period $[0,t)$. We consider the classical risk series (1), or the so-called Cramér-Lundberg series, with the assumption that:

The amount of money collected C(t) is determined and linear over time, that is: $C(t) = ct$, $C > 0$;

String $X = \{X_1, X_2, ...\}$ - Is a random, independent, positive definite string;

The series of intervals between times T_i must pay X_i are also random, independent quantities, then $\tau_i = T_i - T_{i-1}$ is an independent and uniform distribution function;

The cost payment process is a Poisson process with parameter λ, then: $\mathbf{M}N(t) = \lambda t$, $\mathbf{M}X(t) = (\lambda m_1)t$.

Thus, it is crucial for insurance companies to ensure that the capital amount $Z(t)$ always stays above a specific level. Without loss of generality, we assume this level to be zero. The objective is to identify the time of bankruptcy $T = \inf_{t\ge0}\{Z(t) < 0\}$, which is the earliest point when the capital becomes negative. If $\inf\{Z(u), 0 \le u \le t\} \ge 0$ for all $t \ge 0$, then $T = \infty$. Then the probability function of not going bankrupt over an infinite time period $[0,\infty)$ with an initial capital x is defined as (Centeno, 1997):

$$W(x) = \mathbf{P}\{T = \infty \mid Z(0) = x\} = \mathbf{P}\{\sup_{u\ge0}[\sum_{i=1}^{N(u)} X_i - cu] \le x\} \qquad (2)$$

This is one of the characteristics of the Cramer-Lundberg series (1).

When $c - \lambda m_1 > 0$ (condition on net profit of the insurance company) and $F_{X_i}(0) = 0$, the probability function of not going bankrupt of the insurance company satisfies the Volterra - Cramer integral equation as follows (Buhlman, 2007):

$$W(x) - \frac{\lambda}{c}\int_0^x K(x-s)W(s)ds = W(0) \qquad (3)$$

With $K(t) = 1 - F_{X_i}(t)$ - Called the kernel function and the right-hand side is a positive constant $W(0) = c - \lambda m_1 > 0$.

The Volterra equation (3) always has a solution within the class of continuous nuclear functions. The solution method for the aforementioned equation relies on the continuous and bounded nature of the nuclear function K(t).

For example, consider the case where the distribution function of an insurance company's payments is a simple exponential distribution function $F_{X_i}(t) = 1 - e^{-t}$. Then we have:

$K(t) = e^{-t}$; $m_1 = 1$; $W(0) = c - \lambda > 0$

Then equation (3) becomes:

$$W(x) - \frac{\lambda}{c}\int_0^x e^{-(x-s)}W(s)ds = W(0) \Leftrightarrow W(x) = \frac{\lambda}{c}\int_0^x e^{-(x-s)}W(s)ds + W(0) \qquad (4)$$

Solving equation (4) using the Laplace transform method we get:

$$W(P) = v\frac{1}{P+1}W(P) + W_0\frac{1}{P} \Leftrightarrow W(P) = W_0\frac{P+1}{P(P+1-v)}$$

Transforming the Laplace inverse we get:

$$W(x) = \frac{W_0}{1-v}\left[1 - ve^{-(1-v)x}\right] \qquad (5)$$

Examining function $W(x)$, we can determine the probability of the insurance company not going bankrupt according to the amount of initial capital x.

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When performing calculations, we must consider safety parameters for both the insurance company and the reinsurance company. The safety parameters for the insurance company and reinsurance company are denoted as θ and $\xi\ (\xi \geq \theta)$ respectively. Prior to engaging in reinsurance, the insurance premium amount is $c = (1+\theta)\lambda E(X)$. However, when the insurance company participates in reinsurance, the insurance premium amount decreases by a certain value is $c_Y = (1+\theta)\lambda E(X) - (1+\xi)\lambda E(Z)$. Before participating in reinsurance, the distribution function of the size of the insurance company's payments is $F_{X_i}(y)$, after the insurance company participates in reinsurance, the distribution function of the size of the insurance company's payments is $F_{Y_i}(y)$.

Without loss of generality, we denote the distribution function of payment sizes for the insurance company without reinsurance as $F(y)$. When participating in reinsurance, the distribution function becomes $F_X(y)$ and $F_Z(y)$ represents the distribution function for the reinsurance company's payments. Let $W(x)$ be the probability of the insurance company not going bankrupt. Equation (3) determines the probability of non-bankruptcy for the reinsurance model, resulting in the following integral equation:

$$W(x) = \frac{\lambda}{c_Y}\int_0^x K_Y(x-y))W(y)dy + W(0) \qquad (6)$$

With $K_Y(y) = 1 - F_Y(y)$.

The probability equation for an insurance company's non-bankruptcy in reinsurance (6) is essentially identical to that of non-participation (3), they differ only in the distribution function of the size of payments. The distribution function of the size of payments. Therefore, the solution condition and solution method of equation (6) is similar to equation (3) (Gerber, 1997).

4) Evaluation based on Lundberg Exponent:

From the Cramer equation:

$$Z(t) = x + C(t) - \sum_{i=1}^{N(t)} X_i.$$

With in-depth research focusing on assessing the probability of bankruptcy Ψ in insurance and reinsurance problems, Cramer and Lundberg proposed the Lundberg inequality (Cani & Thonhauser, 2016):

$$\Psi(x) < e^{-Rx}$$

And the Lundberg approximation function:

$$\Psi(x) \sim k_{CL}e^{-Rx}, x \to \infty$$

where R is called the Lundberg exponent (also known as the correlation coefficient).

And $k_{CL} = \lim_{x\to\infty}\Psi(x)e^{Rx} = \dfrac{c-\lambda m_1}{\lambda M_X'(R) - c}$ called the Cramer-Lundberg constant.

The Lundberg inequality is extensively utilized in both theoretical and applied research. The application of the Lundberg inequality reveals that the minimization of bankruptcy probability can be achieved by maximizing the Lundberg exponent (Willmost et al., 2016).

B. One-parameter proportional reinsurance model.

Single-parameter proportional reinsurance is a method that divides risks proportionally to the sum insured. The reinsurer guarantees a percentage of each risk based on the insured amount, receives premiums, and is responsible for compensation according to this percentage. This means that the ratio of dividing the insurance amount is equal to the ratio of dividing the insurance premium, as well as the ratio of dividing compensation for losses between the ceding company and the reinsurance company.

Thus we have a mathematical model of one-parameter proportional reinsurance (Pacáková, 2010):

$$X_i = Y_i + Z_i$$

$$Y_i = aX_i$$

$$Z_i = (1-a)X_i$$

Where $X_i (i = 1, 2, ..., N)$ is the claim amount of the risk i and N is the number of losses.

Y_i is the amount retained by the insurance company for risk i.

Z_i is the amount shared by the reinsurer for risk i.

$a\ (0 < a < 1)$ is the division ratio coefficient (parameter).

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Assuming $F(x)$ is the distribution function of the size of original insurance payments of the insurance company X_i, When an insurance company engages in one-parameter proportional reinsurance, the distribution function of the number of payments that the insurance company keeps and transfers to the reinsurer is as follows:

$$F_Y(x) = F(\frac{x}{a})$$

$$F_Z(x) = F(\frac{x}{1-a})$$

In case the insurance company does not participate in reinsurance, the mean value $E(X)$ and variance $D(X)$ of the insurance company's payments are:

$$E(X) = \int_0^{+\infty}(1-F(x))dx$$

$$D(X) = E(X^2) - (E(X))^2 = 2\int_0^{+\infty} x(1-F(x))dx - \left(\int_0^{+\infty}(1-F(x))dx\right)^2$$

In the case of Insurers engaging in one-parameter proportional reinsurance, the insurer's expectations are divided into two components, the average value of the portion retained by the insurer $E(Y)$, and the portion transferred to the reinsurance company $E(Z)$:

$$E(Y) = a\int_0^{+\infty}(1-F(x))dx$$

$$E(Z) = (1-a)\int_0^{+\infty}(1-F(x))dx$$

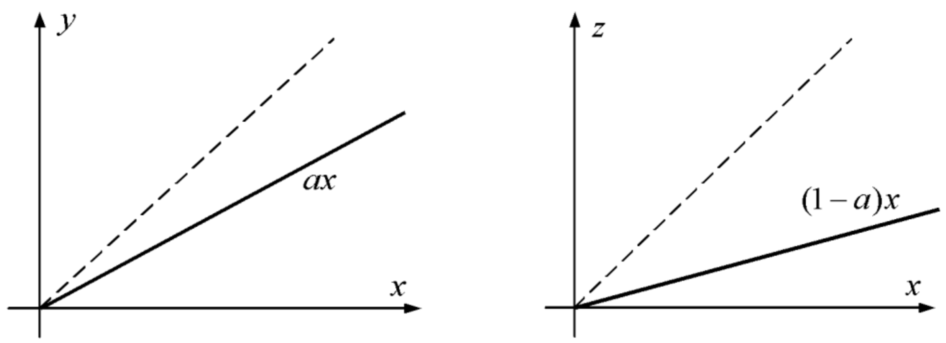
Variance of the portion retained by the insurance company $D(Y)$ and the portion transferred to reinsurer $D(Z)$:

$$D(Y) = 2a^2\int_0^{+\infty} x(1-F(x))dx - \left(a\int_0^{+\infty}(1-F(x))dx\right)^2$$

$$D(Z) = 2(1-a)^2\int_0^{+\infty} x(1-F(x))dx - \left((1-a)\int_0^{+\infty}(1-F(x))dx\right)^2$$

Without loss of generality, we consider X, Y and Z to be the payment at time i, then the combination of payments in the one-parameter proportional model is shown in the following figure:

Figure 1. Risk allocation in proportional reinsurance.



(a) Risks of the insurance company *(b) Risks of the company reinsurer*

III. MATERIALS AND METHODS

A. Risk optimization in reinsurance.

To mitigate risks, individuals and organizations employ a reliable approach: insurance - the transfer of risks to insurance companies. Nevertheless, insurance companies themselves may face risks that necessitate safeguarding. Consequently, insurance companies also seek protection, which takes the form of reinsurance. Reinsurance is a crucial stage in the business cycle, allowing for the distribution of risks and ensuring the continuity and viability of both insurance companies and the insurance market (Schmidli, 2002a).

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For company insurance and reinsurance, maintaining capital above a specific threshold is crucial. The term "bankruptcy time" refers to the earliest point at which the capital amount falls below this threshold. The probability of this happening (bankruptcy) within a finite period of time $[0,t)$ given the initial capital x is called the probability of bankruptcy and is denoted by $\psi(x,t)$. Thus, optimizing risk in reinsurance is finding a reinsurance operation plan that minimizes the probability of bankruptcy $\psi(x,t)$ (Buhlman, 2007).

B. Cramer equation and the Lundberg exponent R.

1) The Cramér-Lundberg model:

Dynamic models are distinct from static models as they account for events that unfold over time. The most basic form of such models encompasses two processes: the collection of insurance premiums and the payment of insurance claims. We make the assumption that insurance policies are received continuously and that the flow of premiums is proportional to the duration of any given period. Consequently, we arrive at the following dynamic model for the insurance portfolio (Hald & Schmidli, 2004).

$$V(t) = x + ct - \sum_{i=1}^{N(t)} X_i, \qquad (7)$$

In which, $x = V(0)$ – Initial capital.

ct – The insurance proceeds obtained are proportional over time with the coefficient $c > 0$.

$N(t)$ - Number of payments in time period $[0,t)$, is a Poisson process with parameters λ.

String $X = \{X_1, X_2, ...\}$ – is a random, independent, positive definite string.

The risk process (7) is called the Cramér-Lundberg model, or the classical risk model.

The symbol $F(x)$ is the risk distribution function of X_i, $m_k = \mathbf{M}X_1^k$ is its moments. $g_X(t) = \mathbf{M}e^{tX}$ is the exponential function at each time point of the random variable X, combined with the random variable X_1. We see that, when $g_X(t) < \infty$, all times $m_k = \mathbf{M}X^k$ exist.

2) Probability of bankruptcy:

For insurance companies, it is important that the amount of capital $V(t)$ at all times remains above a certain level, without loss of generality, we assume that this level is zero (Belkina et al., 2014). The problem is determining the time of bankruptcy $T = \inf\{t > 0 : V(t) < 0\}$, that is, the earliest time in interval $[0,\infty)$ when capital becomes negative. Period $[0,T]$ is called the prosperous period of the insurance company. If $\sup_{0 \le u \le t} V(u) > 0$ for all $t \ge 0$ then $T = \infty$.

Value $\Psi(x,t) = \mathbf{P}\{T \le t \mid V(0) = x\}$ is called the probability of bankruptcy in a finite time period $[0,t)$ with initial capital X. And value

$$\Psi(x) = \lim_{t\to\infty} \Psi(x,t) = \mathbf{P}\{T < \infty \mid V(0) = x\}$$

is called the probability of bankruptcy in an infinite time period $[0,\infty)$ with initial capital x. It's easy to see that, $\Psi(x,t)$ becomes smaller as X gets larger, and $\Psi(x,t)$ becomes larger as t increases.

Symbol T_i - The time of receipt of the request, $T_0 = 0$. Put $Y_i = c(T_i - T_{i-1}) - X_i$. If we just look at the process at the time the request is received, it can be seen $V(T_k) = x + \sum_{i=1}^{k} Y_i$ – random walk, then $\Psi(x) = \mathbf{P}\{\inf_{k\in N} V(T_k) < 0\}$. From contingency theory, bankruptcy occurs if and only if $\mathbf{M}Y_i \le 0$. Therefore we can see that:

$$\mathbf{M}Y_i > 0 \Leftrightarrow \frac{c}{\lambda} - m_1 > 0 \Leftrightarrow c > \lambda m_1 \Leftrightarrow \mathbf{M}[V(t) - V(0)] > 0.$$

Easy to see $\mathbf{M}[V(t) - V(0)] = (c - \lambda m_1)t$. Condition:

$$c - \lambda m_1 > 0 \qquad (8)$$

can be understood as an excess condition of the total amount of premiums received over the total amount paid in the interval $[0,t)$, also known as the condition of having a net profit.

If condition (8) occurs, then $\lim_{x\to\infty} \Psi(x) = 0$.

For a net profit situation to occur, we set: $c = (1+\theta)\lambda m_1, \quad \theta > 0.$

Afterward $\mathbf{M}[V(t) - V(0)] = \theta\lambda m_1 t > 0$. Thus, risk insurance premium θ is a prerequisite for the operation of insurance companies.

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Cani & Thonhauser (2016) showed, with net profit conditions $c-\lambda m_1>0$ and $F(0)=0$, probability of not going bankrupt $W(x))=1-\Psi(x)$ in model (7) the integral equation is met:

$$W(x)=\frac{\lambda}{c}\int_0^x K(x-s)W(s)ds+W(0) \quad (9)$$

with $K(t)=1-F(t)$ called the kernel function and constant $W(0)=c-\lambda m_1>0$.

Equation (9) always has a solution in the class of integrable functions.

The solution belongs to a class $C(0,\infty)$ continuous functions are given by the kernel variation limit. For example, it is enough to make kernels $K(x,s)=K(x-s,0)\equiv K(x-s)$ is limited in magnitude and has a finite number of discontinuities on the horizontal line $t=x-s$.

The equation for the probability of bankruptcy according to (9) has the form:

$$cW'(x)=\lambda\{W(x)-\int_0^x W(x-y)dF(y)\}.$$

With exponential distribution $X\sim Exp(\mu)$ (9) can be solved clearly as follows:

$$W(x)=1-\frac{\lambda}{\mu c}e^{-(\mu-\lambda/c)x}.$$

3) Correlation coefficient equation:

Consider the classical hazard model (with Poisson distribution) with Cramer conditional assumptions to ensure the solution of the equation $R>0$, We have:

$$\lambda M_X(R)=\lambda+cR \quad (10)$$

With:

$M_X(R)=\mathbf{M}e^{RX}=\int_0^\infty e^{Rx}dF(x)$ - moment function between X and distribution function $F(x)$ R is called the correlation coefficient, or the Lundberg exponent.

According to (10), we have:

$$\lim_{x\to\infty}\Psi(x)e^{Rx}=\mathrm{k}_{\mathrm{CL}}$$

With Cramér-Lundberg constant:

$$\mathrm{k}_{\mathrm{CL}}=\frac{\int_0^\infty e^{Ru}du\int_u^\infty(1-F(v))dv}{\int_0^\infty ue^{Ru}(1-F(u))du} \quad (10a)$$

Then, (10) can be expressed in an approximate form as follows:

$$\Psi(x)\sim k_{CL}\,e^{-Rx},\ x\to\infty \quad (10b)$$

Example 1: Give $X\sim Exp(\mu)$ it mean $F(x)=1-e^{-\mu x}$. Then we have the probability of bankruptcy:

$$\Psi(x)=L^{-1}e^{-Rx}=(\lambda/c\mu)e^{-(\mu-\lambda/c)x}$$

Thus:

$$k_{CL}=L^{-1}=\frac{\lambda}{c\mu}=\frac{1}{1+\theta},\ R=\mu-\lambda/c=\frac{\mu\theta}{1+\theta}$$

Based on the Lundberg inequality, we have:

$\Psi(x)\le e^{-Rx}$ với $x\ge 0$

And:

$\Psi(x)\sim k_{CL}e^{-Rx}$ khi $x\to\infty$

4) The relationship between the Lundberg exponent and bankruptcy probability:

The Lundberg exponent R used in the theory of risk assessment to determine the probability of bankruptcy $\psi(x)$. In general, it is difficult to calculate accurately. So we will define R based on its upper and lower limits.

Theorem 1. Suppose $\mathbf{M}e^{tX}<\infty$ and exist the Lundberg exponent (correlation coefficient) R then we have the following assertion:

(1) $R<\frac{2(c-\lambda m_1)}{\lambda m_2}$;

(2) If $suppF=[0,M]$, with $M<\infty$, then $R>\frac{1}{M}\log\frac{c}{\lambda m_1}$;

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(3) $\Psi(x) < e^{-Rx}$;

(4) $\Psi(x) \sim k_{CL} e^{-Rx}, x \to \infty$, *với:*

$$k_{CL} = \lim_{x\to\infty} \Psi(x) e^{Rx} = \frac{c - \lambda m_1}{\lambda M_X'(R) - c} \quad (11)$$

Statement (3) in Theorem 1, known as the Lundberg inequality, is widely utilized in both theoretical and applied research. The importance of maximizing the R value in reinsurance becomes evident through the application of the Lundberg inequality.

Statement (4) in Theorem 1, also referred to as the Cramér-Lundberg approximation, serves as the fundamental basis in risk theory. Its practical implementation yields highly favorable outcomes, which will be further explored in the algorithms section focusing on determining the correlation coefficient R. This clearly illustrates that risk optimization can be achieved by maximizing the R value (Belkina et al., 2014).

IV. RESULT

A. Optimize risk with a proportional reinsurance model based on maximizing the Lundberg exponent.

In this section we consider calculating the Lundberg exponent $\mathbf{R}$ according to the coefficient *a* of the proportional reinsurance model, thereby determining the probability of bankruptcy.

Assume the distribution function is exponential $F(x) = 1 - e^{-\mu x}$.

From that: $dF(x) = \mu e^{-\mu x} dx$, $\mathbf{M}X = \frac{1}{\mu}$

$F_Y(x) = 1 - e^{-\mu x/a}$, $dF_Y(x) = \frac{\mu}{a} e^{-\mu x/a} dx$, $\mathbf{M}Y = \frac{a}{\mu}$

With reinsurance company, $\mathbf{Z} = (1-a)X$

$F_Z(x) = 1 - e^{-\mu x/(1-a)}$, $dF_Z(x) = \frac{\mu}{1-a} e^{-\mu x/(1-a)} dx$, $\mathbf{M}Z = \frac{1-a}{\mu}$

Net income from reinsurance:

$$c_Y = (1+\theta)\lambda \mathbf{M}X - (1+\xi)\lambda \mathbf{M}Z$$

$$= \frac{\lambda}{\mu}\left[(1+\xi)a - (\xi - \theta)\right]$$

According to (10) we have:

$$\lambda \int_0^\infty e^{Rx} dF_Y(x) = \lambda + c_Y R$$

$$\frac{\mu}{a}\int_0^\infty e^{(R-\mu/a)x} dx = 1 + \frac{1}{\mu}\left[(1+\xi)a - (\xi-\theta)\right]R$$

$$R = \frac{\mu}{a}\left[\frac{\xi a - (\xi - \theta)}{(1+\xi)a - (\xi-\theta)}\right]$$

Basic dependencies:

$$R = \frac{\mu}{a}\left[\frac{\xi a - (\xi - \theta)}{(1+\xi)a - (\xi-\theta)}\right], \quad 0.5 < a \le 1$$

1) Case $\xi = \theta$

$$R = \frac{\mu\theta}{a(1+\theta)}, \ 0 < a < 1.$$

Example 2: Suppose we have $\xi = \theta = 0.1, \ \mu = 0.1$. Then:

$$R = \frac{1}{110a}, \ 0 < a < 1$$

The results are shown as follows.

Figure 2. Value of *R* according to a with $\xi = \theta = 0.1, \ \mu = 0.1$

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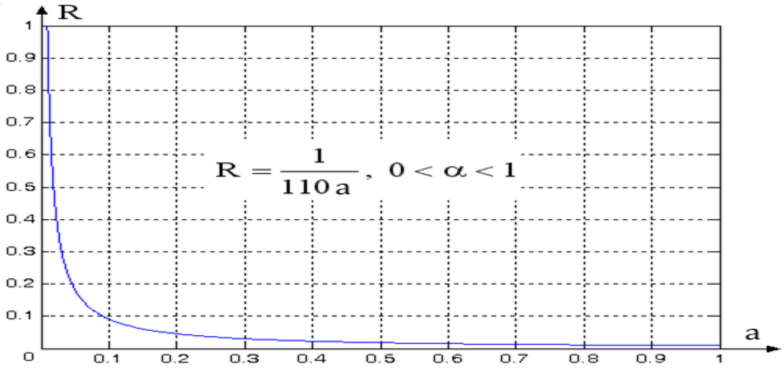


(Source: Calculated from research results)

2) Case $\xi > \theta$

$$R = \frac{\mu}{a}\left[\frac{\xi a - (\xi - \theta)}{(1+\xi)a - (\xi - \theta)}\right], \quad 0.5 < a \le 1$$

The optimal value of *a*:

$$a_{1,2} = \frac{(1+\xi) \pm \sqrt{(1+\xi)}}{\xi(1+\xi)}(\xi - \theta)$$

Example 3: Suppose we have $\theta = 0.1$, $\xi = 0.2$, $\mu = 0.1$. Then:

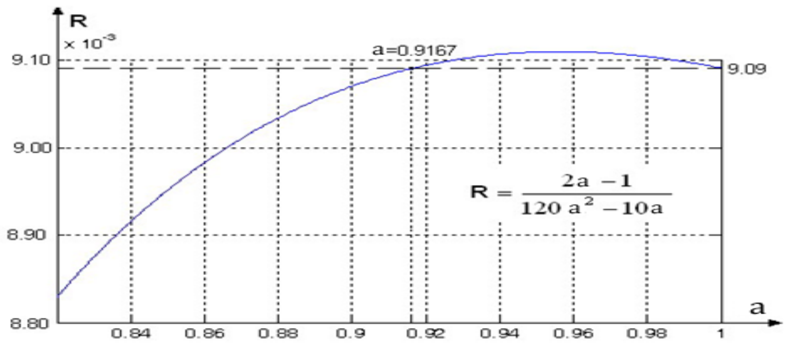
$$a_1 = 0.0436, \; a_2 = 0.9564$$

$$R = \frac{2a - 1}{120a^2 - 10a}$$

Once you have the results of *R* we can easily calculate it k_{CL} and $\psi(x)$ by formulas (10a) and (10b).

The results are shown as follows.

Figure 3. Value of *R* according to a with $\theta = 0.1$, $\xi = 0.2$, $\mu = 0.1$



(Source: Calculated from research results)

B. Algorithm to determine the Lundberg exponent R.

In the individual risk model, the concept of limit reinsurance is relatively straightforward. By employing a risk distribution function, statistical modeling can yield the desired outcome. This approach is particularly effective for exponential and uniform distributions. However, for other scenarios, R maximization is achieved through approximate calculations utilizing specialized software.

The Lundberg inequality suggests that the correlation coefficient is a risk measure. Hence, to reduce risk, it is important to increase its value. Hard and Shmidli (2004) have also researched these matters. The algorithm for maximizing the Lundberg exponent is as follows:

Algorithm 1. Risk sharing method*:* $X = Y + Z$, with $Y = aX$, $Z = (1-a)X$

Input data and symbols:

$g_X(r) = \mathbf{M}e^{rX}$ *– Exponential moment;*

$r_\infty := \sup\{r : g_X(r) < \infty\} > 0$;

(θ, ξ) *– Risky situation;*

(a_0, R) *– Function at its maximum point* $R = R(a)$

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Order of steps:

(1) $\rho :=$ Original value of the equation $g'_X(r) = (1+\xi)m_1$, if it exists;

(2) $\rho := r_\infty$, If the original value of the equation is in (1) does not exist;

(3) $a'_0 = \frac{(\xi-\theta)m_1\rho}{(1+\xi)m_1\rho+1-g_X(\rho)}$;

(4) $a_0 = \min\{a'_0, 1\}$;

(5) If $a_0 = a'_0$, then $R(a_0) := \rho/a_0$.

If the browser $g'_X(r) = (1+\xi)m_1$ It is not easy to solve, you can use the approximate method.

We can also study the maximization of R based on the Markov risk model. Using the Markov risk model, assumption $X \sim Exp(\mu)$, mean $F(x) = 1 - e^{-\mu x}$, and the input is a Poisson process. We have a direct consequence of algorithm 2 described as follows

Algorithm 2. Risk sharing method: $X = Y + Z$, với $Y = aX$, $Z = (1-a)X$

Input data and symbols:

$X \sim Exp(\mu)$, mean $F(x) = 1 - e^{-\mu x}$;

(θ, ξ) – *Risky situation;*

(a_0, R) – *Function at its maximum point* $R = R(a)$

Order of steps:

(1) If $\xi = \theta$, then $a_0 R := \frac{\mu\theta}{1+\theta}$, $0 < a_0 \le 1$;

(2) If $0 < \xi - \theta \le \theta(1+\theta)$, then

$$a_0 := \frac{\xi - \theta}{\sqrt{1+\xi}(\sqrt{1+\xi} - 1)}, \quad \frac{R}{\mu} := \frac{(\sqrt{1+\xi} - 1)^2}{\xi - \theta}, \tag{12}$$

with $a_0 \in (\beta, 1)$, and $\beta = \frac{(1+\theta)(\xi-\theta)}{\theta(1+\xi)}$;

(3) If $\xi - \theta \ge \theta(1+\theta)$, then $a_0 := 1$, $R := \frac{\mu\theta}{1+\theta}$.

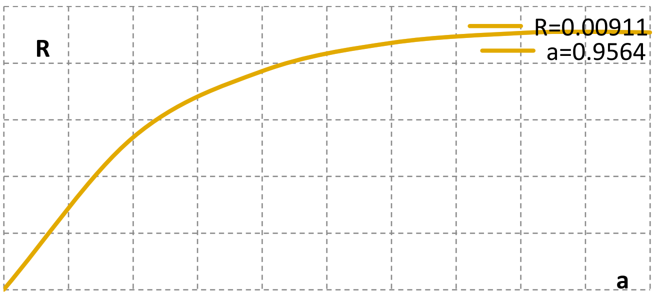
Example 4. Return to example 3. Suppose $X \sim Exp(\mu)$, $\theta = 0.1$, $\xi = 0.2$, $\mu = 0.1$. We know that to maximize the R value, a needs to be considered in the range $[\frac{1}{2}, 1]$. According to (10), we can determine $a = 0.9564$.

Calculate the R coefficient according to a:

$$R = \frac{2a - 1}{120a^2 - 10a}.$$

The dependence on a of R is shown in Figure 4. With the given values of θ, ξ and μ we can also calculate $\beta = \frac{11}{12} = 0.9167$. And looking at Figure 4, we see that the meaning of maximizing the R value is to keep $a = 0.9564$.

Figure 4. The value of R according to the scaling factor a.



(Source: Calculated from research results)

C. Risk assessment is based on maximizing the Lundberg exponent R.

We will analyze the value R, the Cramér-Lundberg constant k_{CL}, and the probability of bankruptcy Ψ based on the scaling factor a when the distribution function of payment sizes follows an exponential distribution. Let's assume $\theta = 0.4$, $\xi = 0.5$ as the initial capital amount in calculation units $x = 1$.

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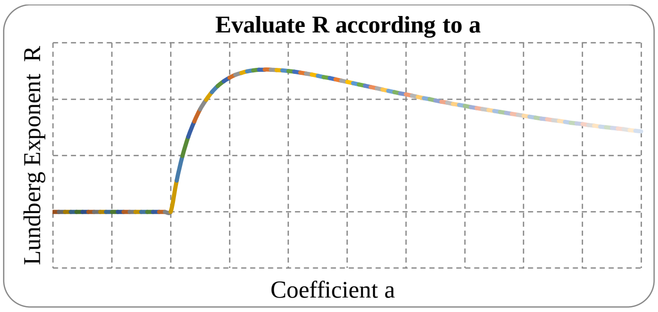
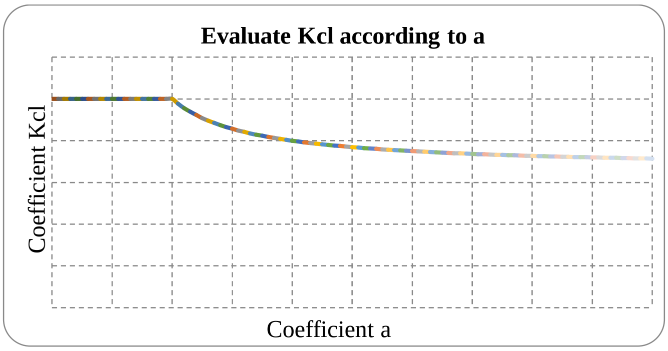
Figure 5. Express the value of R in terms of the scaling factor a



(Source: Calculated from research results)

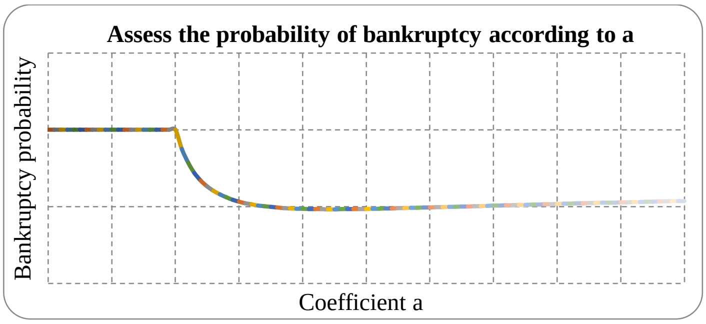
Looking at Figure 5, we see that with the distribution function of the size of insurance payments being an exponential distribution function, R reaches the value max = 0.50505 at the scaling factor a = 0.36

Figure 6. Express the value of k_{CL} in terms of the scaling factor a



(Source: Calculated from research results)

Figure 7. Express the value of Ψ in terms of the scaling factor a



(Source: Calculated from research results)

Looking at Figure 7, we see that with the distribution function of the size of insurance payments being an exponential distribution function, ψ reaches the value min = 0.48277 at the scale coefficient a = 0.45.

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